Int. J. Solids Structures Vol. 14, pp. 971–986
© Pergamon Press Ltd., 1978. Printed in Great Britain

0020-7683/78/1201-0971/$02.00/0

THE FUNDAMENTAL SOLUTION IN THE THEORY OF SHALLOW SHELLS†

TETSUYA MATSUI‡ and OSAMU MATSUOKA§

Department of Architecture, Nagoya University, Nagoya, Japan

(Received 12 October 1977; in revised form 30 March 1978)

Abstract—A series representation for the fundamental solution of the shallow shell equations is obtained by means of a plane-wave decomposition of the Dirac δ-function. From this solution we can produce the singular solutions which correspond to concentrated forces, couples and thermal hot spots applied to a shallow shell with an arbitrary quadratic middle surface. The solutions converge for the entire range of the Gaussian curvature. Numerical results are presented for the case of a concentrated normal force acting on infinite shells having positive, zero or negative Gaussian curvature.

INTRODUCTION

The calculation of the stresses and deflections in the local environment of concentrated loads or concentrated areas of heating on a thin shell is a problem of important concern to the shell designer. If a concentrated force or a thermal hot spot is applied to a thin shell on its sufficiently smooth surface, then the disturbances of the stresses and deflections caused by the concentrated force or the hot spot will be restricted within a localized region around the loaded point. Since almost any shell has a small slope within the restricted region, it is reasonable to apply the results from the shallow shell theory to the problem of more general shells. Thus the problem of concentrated loading or concentrated area of heating on a shallow shell with an arbitrary quadratic middle surface is one of fundamental importance.

Since the pioneer work of Reissner[1] for a spherical cap subjected to a concentrated normal force, many investigators have made significant contributions to the development and identification of the singular solutions which represent concentrated loading and heating applied to a shallow shell. Chernyshev[2] has proved that the dominant parts of the stresses and deflections in the neighbourhood of a concentrated force or couple on a general shell are the same as for the corresponding problem of the flat-plate. Lukasiewicz[3] has presented detailed results for a shallow spherical cap subjected to concentrated normal and tangential forces and concentrated couples. Flügge and Conrad[4] have obtained the singular solutions for thermal hot spots applied to a cylindrical shell. Sanders and Simmonds[5] have presented the solutions for concentrated normal and tangential forces applied to a shallow cylindrical shell. Foresberg and Flügge[6] have investigated the solution for a normal point load applied to a shallow elliptic paraboloid. Flügge and Elling[7] have developed the solutions for a concentrated normal force and thermal hot spots applied to a shallow shell with an arbitrary quadratic middle surface, and Elling[8] has extended the solutions to present the results for concentrated tangential forces and concentrated couples. Sanders[9] has attempted a unified treatment of the singular solutions to the shallow shell equations by means of a Fourier transform, but the inverse transforms are difficult to obtain explicitly except in the case of a sphere[3] and a cylinder[5]. For general shells, only approximate solutions of the inverse transforms have been developed by Lukasiewicz[10].

The problem of concentrated loading or concentrated area of heating on a shallow shell with an arbitrary quadratic middle surface can be reduced to the problem of finding the fundamental solution of the shallow shell equations. In the present paper, a series representation for this fundamental solution is obtained by means of a plane-wave decomposition of the Dirac δ-function[11, 12]. From this solution we can produce the singular solutions which represent

†The present paper is part of a dissertation submitted by the first author to Kyoto University in partial fulfilment of the requirements for the degree of Doctor of Engineering.
‡Associate professor.
§Professor.

concentrated forces, couples and thermal hot spots applied to the shallow shell. The identical solutions have already been obtained by Flügge and Elling[7] and Elling[8] by use of the separation of variable method in polar co-ordinates. However, their solutions are not entirely satisfactory. The solutions have certainly the correct singularities for the concentrated loading or heating, but they include components which grow exponentially at infinity. Flügge and Elling have suggested to suppress such an unwanted growth by fulfilling the specified boundary conditions at the outer edges of the finite shell, but it requires rather troublesome algebraic manipulation. The objective herein is to seek the solution which is well-behaved at infinity, i.e. to construct Green's function for an infinite shell. Numerical results are presented for the case of a concentrated normal force acting on infinite shells having positive, zero or negative Gaussian curvature, and comparisons are made to study the influence of shell geometry on the distributions of stresses and deflections.

BASIC EQUATIONS

We will restrict our investigation to a shallow shell with a quadratic middle surface expressed by

$$z = \frac{1}{2a}x^2 + \frac{1}{2b}y^2 \tag{1}$$

where a and b correspond to the two principal radii of curvature of the shell. Figure 1 shows the basic co-ordinate system used in this expression.

The basic partial differential equations which govern the behaviour of this shallow shell can be reduced to a set of equations for three displacement functions Φ_1, Φ_2 and Φ_3. The equations are[13]

$$\begin{aligned} L(\Phi_1) &= -\frac{X}{K} - \frac{12}{t^2}(1+\nu)\bar{\alpha}\frac{\partial T}{\partial x} \\ L(\Phi_2) &= -\frac{Y}{K} - \frac{12}{t^2}(1+\nu)\bar{\alpha}\frac{\partial T}{\partial y} \\ L(\Phi_3) &= \frac{Z}{K} - \frac{1+\nu}{t}\bar{\alpha}\Delta\bar{T} \end{aligned} \tag{2}$$

where

$$L = \Delta\Delta\Delta\Delta + \frac{12}{t^2}(1-\nu^2)\left(\frac{1}{b}D_1{}^2 + \frac{1}{a}D_2{}^2\right)^2 \tag{3}$$

with

$$D_1 = \frac{\partial}{\partial x},\quad D_2 = \frac{\partial}{\partial y},\quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4}$$

and

$$K = \frac{Et^3}{12(1-\nu^2)},\ \text{bending stiffness}$$

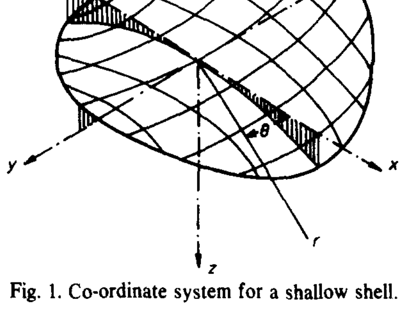

Fig. 1. Co-ordinate system for a shallow shell.

X, Y, Z = distributed loads applied in x, y, z directions, respectively, T = average temperature over a section of shell, $\bar{T}$ = temperature difference between inner and outer shell surfaces, $\bar{\alpha}$ = coefficient of thermal expansion, ν = Poisson's ratio, t = thickness of shell, E = Young's modulus.

Let Φ be the fundamental solution of the differential operator L, i.e. the solution which satisfies the inhomogeneous differential equation

$$L(\Phi) = \delta(x, y) \tag{5}$$

where $\delta(x, y)$ denotes the Dirac δ-function. Then the particular solutions to the set of differential equations (2) for arbitrary loading and temperature distribution can be expressed in the form

$$\left.\begin{aligned} \Phi_1 &= -\int\int \Phi(x-\xi, y-\eta)\left\{\frac{X(\xi,\eta)}{K} + \frac{12}{t^2}(1+\nu)\bar{\alpha}\frac{\partial T(\xi,\eta)}{\partial \xi}\right\} \mathrm{d}\xi\,\mathrm{d}\eta \\ \Phi_2 &= -\int\int \Phi(x-\xi, y-\eta)\left\{\frac{Y(\xi,\eta)}{K} + \frac{12}{t^2}(1+\nu)\bar{\alpha}\frac{\partial T(\xi,\eta)}{\partial \eta}\right\} \mathrm{d}\xi\,\mathrm{d}\eta \\ \Phi_3 &= \int\int \Phi(x-\xi, y-\eta)\left\{\frac{Z(\xi,\eta)}{K} - \frac{1+\nu}{t}\bar{\alpha}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\bar{T}(\xi,\eta)\right\} \mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned}\right\} \tag{6}$$

Once the set of differential equations (2) is solved, the displacements can be expressed in terms of the displacement functions as follows:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} \tag{7}$$

where

$$\left.\begin{aligned} l_{11} &= \frac{t^2}{12}\left(D_1^2 + \frac{2}{1-\nu}D_2^2\right)\Delta\Delta + \left(\frac{1}{a^2} + \frac{2\nu}{ab} + \frac{1}{b^2}\right)D_1^2 + \frac{2(1+\nu)}{a^2}D_2^2 \\ l_{12} &= l_{21} = -\frac{1+\nu}{1-\nu}\frac{t^2}{12}D_1D_2\Delta\Delta - \left(\frac{1}{a} - \frac{1}{b}\right)^2 D_1D_2 \\ l_{22} &= \frac{t^2}{12}\left(\frac{2}{1-\nu}D_1^2 + D_2^2\right)\Delta\Delta + \frac{2(1+\nu)}{b^2}D_1^2 + \left(\frac{1}{a^2} + \frac{2\nu}{ab} + \frac{1}{b^2}\right)D_2^2 \\ l_{13} &= l_{31} = \left(\frac{1}{a} + \frac{\nu}{b}\right)D_1^3 + \left(\frac{2+\nu}{a} - \frac{1}{b}\right)D_1D_2^2 \\ l_{23} &= l_{32} = \left(\frac{1}{b} + \frac{\nu}{a}\right)D_2^3 + \left(\frac{2+\nu}{b} - \frac{1}{a}\right)D_1^2D_2 \\ l_{33} &= \Delta\Delta. \end{aligned}\right\} \tag{8}$$

The stress resultants can be related to the displacements and temperature as follows:

$$\left.\begin{aligned} N_x &= \frac{Et}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - \left(\frac{1}{a} + \frac{\nu}{b}\right)w - (1+\nu)\bar{\alpha}T\right] & M_y &= -K\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} + \frac{1+\nu}{t}\bar{\alpha}\bar{T}\right) \\ N_y &= \frac{Et}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x} - \left(\frac{1}{b} + \frac{\nu}{a}\right)w - (1+\nu)\bar{\alpha}T\right] & M_{xy} &= -(1-\nu)K\frac{\partial^2 w}{\partial x \partial y} \\ N_{xy} &= \frac{Et}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & Q_x &= -K\frac{\partial}{\partial x}\left(\Delta w + \frac{1+\nu}{t}\bar{\alpha}\bar{T}\right), \\ M_x &= -K\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \frac{1+\nu}{t}\bar{\alpha}\bar{T}\right) & Q_y &= -K\frac{\partial}{\partial y}\left(\Delta w + \frac{1+\nu}{t}\bar{\alpha}\bar{T}\right). \end{aligned}\right\} \tag{9}$$

The positive directions of the stress resultants and displacements are shown in Fig. 2.

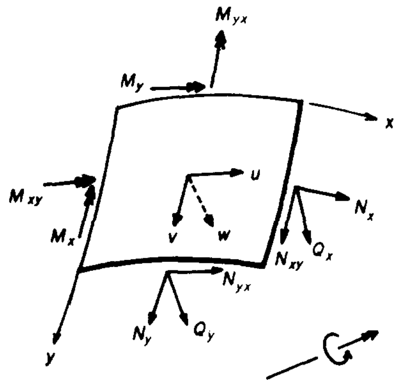

Fig. 2. Sign conventions for stress resultants and displacements.

CONCENTRATED FORCES AND THERMAL HOT SPOTS

The purpose of the present paper is to study the particular solutions to the set of differential equations (2), when concentrated forces, couples or thermal hot spots are applied to the shallow shell. An effective approach to this goal is to use the Dirac δ-function to describe the concentrated forces, couples, etc. and then to solve the inhomogeneous shell equations by appealing to the theory of distributed functions. The problem is, then, reduced to finding the fundamental solution of eqn (5).

Consider now the case in which concentrated forces, couples or hot spots are applied to the shell at a single point. In the present case, there is no loss of generality if the origin is taken as the loaded point. In the following lists are given for the concentrated forces, couples and hot spots described in terms of the Dirac δ-function and their corresponding solutions:

1. Tangential force in x-direction

$$\{X, Y, Z\} = \{P_x\delta(x, y), 0, 0\}, \qquad \{\Phi_1, \Phi_2, \Phi_3\} = \left\{-\frac{P_x}{K}\Phi(x, y), 0, 0\right\}.$$

2. Tangential force in y-direction

$$\{X, Y, Z\} = \{0, P_y\delta(x, y), 0\}, \qquad \{\Phi_1, \Phi_2, \Phi_3\} = \left\{0, -\frac{P_y}{K}\Phi(x, y), 0\right\}.$$

3. Normal force

$$\{X, Y, Z\} = \{0, 0, P_z\delta(x, y)\}, \qquad \{\Phi_1, \Phi_2, \Phi_3\} = \left\{0, 0, \frac{P_z}{K}\Phi(x, y)\right\}.$$

4. Tangential couple in x-direction

$$\{X, Y, Z\} = \left\{0, 0, \bar{M}_x\frac{\partial\delta(x, y)}{\partial x}\right\}, \qquad \{\Phi_1, \Phi_2, \Phi_3\} = \left\{0, 0, \frac{\bar{M}_x}{K}\frac{\partial\Phi(x, y)}{\partial x}\right\}.$$

5. Tangential couple in y-direction

$$\{X, Y, Z\} = \left\{0, 0, \bar{M}_y\frac{\partial\delta(x, y)}{\partial y}\right\}, \qquad \{\Phi_1, \Phi_2, \Phi_3\} = \left\{0, 0, \frac{\bar{M}_y}{K}\frac{\partial\Phi(x, y)}{\partial y}\right\}.$$

6. Normal couple

$$\{X, Y, Z\} = \left\{-\frac{\bar{M}_z}{2}\frac{\partial\delta(x, y)}{\partial y}, \frac{\bar{M}_z}{2}\frac{\partial\delta(x, y)}{\partial x}, 0\right\}$$

$$\{\Phi_1, \Phi_2, \Phi_3\} = \left\{\frac{\bar{M}_z}{2K}\frac{\partial\Phi(x, y)}{\partial y}, -\frac{\bar{M}_z}{2K}\frac{\partial\Phi(x, y)}{\partial x}, 0\right\}.$$

7. Plane hot spot

$$\{T, \bar{T}\} = \{\mu\delta(x, y), 0\}$$

$$\{\Phi_1, \Phi_2, \Phi_3\} = \left\{ -\frac{12}{t^2}(1+\nu)\bar{\alpha}\mu \frac{\partial \Phi(x, y)}{\partial x}, -\frac{12}{t^2}(1+\nu)\bar{\alpha}\mu \frac{\partial \Phi(x, y)}{\partial y}, 0 \right\}.$$

8. Bending hot spot

$$\{T, \bar{T}\} = \{0, \bar{\mu}\delta(x, y)\}$$

$$\{\Phi_1, \Phi_2, \Phi_3\} = \left\{0, 0, -\frac{1+\nu}{t}\bar{\alpha}\bar{\mu}\Delta\Phi(x, y)\right\}. \tag{10}$$

FUNDAMENTAL SOLUTION

Plane-wave representation for Φ

Equation (5) is now written out in detail in the form

$$L(\Phi) = \left[\Delta\Delta\Delta\Delta + \left(\alpha^2 \frac{\partial^2}{\partial x^2} + \beta^2 \frac{\partial^2}{\partial y^2}\right)^2\right]\Phi = \delta(x, y) \tag{11}$$

where

$$\alpha^2 = \frac{\sqrt{(12(1-\nu^2))}}{tb}, \qquad \beta^2 = \frac{\sqrt{(12(1-\nu^2))}}{ta}. \tag{12}$$

In the subsequent development of the fundamental solution of eqn (11), we follow the work of Gel'fand and Shilov[11, pp. 122–124]. We first replace the δ-function on the r.h.s. of eqn (11) by

$$\frac{2r^\lambda}{\Omega_2\Gamma\left(\frac{\lambda+2}{2}\right)} \tag{13}$$

which is equal to the δ-function for $\lambda = -2$ [11, *p*. 74, eqn (9)], where Ω_2 is the surface area of the unit sphere in the two-dimensional space, Γ denotes the Gamma function and $r = \sqrt{(x^2 + y^2)}$. By expanding eqn (13) into plane-waves [11, p. 76, eqn (4)], we obtain the equation

$$L(\Phi) = \frac{1}{\Omega_2\pi^{1/2}\Gamma\left(\frac{\lambda+1}{2}\right)} \int_\Omega |\omega_1 x + \omega_2 y|^\lambda \, d\Omega \tag{14}$$

where (ω_1, ω_2) is the co-ordinate of a point on the unit sphere and $d\Omega$ is the surface element on the unit sphere. If we now solve

$$L(\Phi) = \frac{1}{\Omega_2\pi^{1/2}\Gamma\left(\frac{\lambda+1}{2}\right)} |\omega_1 x + \omega_2 y|^\lambda \tag{15}$$

for ϕ, a function depending only on $\rho = \omega_1 x + \omega_2 y$, we can write the solution to eqn (14) in the form

$$\Phi = \int_\Omega \phi(\omega_1 x + \omega_2 y) \, d\Omega. \tag{16}$$

Equation (16) is called the plane-wave representation for the fundamental solution. Now the partial differential operators can be written as

$$\frac{\partial}{\partial x} = \omega_1 \frac{d}{d\rho}, \qquad \frac{\partial}{\partial y} = \omega_2 \frac{d}{d\rho}. \tag{17}$$

When these are applied to eqn (15), we can obtain the ordinary differential equation of eighth order

$$\frac{d^8\phi}{d\rho^8}+p^4\frac{d^4\phi}{d\rho^4}=\frac{1}{\Omega_2\pi^{1/2}\Gamma\left(\frac{\lambda+1}{2}\right)}|\rho|^\lambda \tag{18}$$

for ϕ, where

$$p^2=\alpha^2\omega_1{}^2+\beta^2\omega_2{}^2. \tag{19}$$

The solution to eqn (11) now reduces to solving the ordinary differential equation (18). After four times integration of eqn (18) and setting $\lambda\rightarrow-2$, we obtain [11, p. 55, eqn (4)]

$$\frac{d^4\phi}{d\rho^4}+p^4\phi=\frac{1}{8\pi^2}\rho^2\log|\rho|. \tag{20}$$

The solution to eqn (20) that vanishes at infinity can be obtained in a standard fashion by applying the variation of parameter method[14]. The solution is

$$\phi=f_1e^{\kappa\rho(1+i)}+f_2e^{\kappa\rho(1-i)}+f_3e^{-\kappa\rho(1+i)}+f_4e^{-\kappa\rho(1-i)} \tag{21}$$

where

$$\left.\begin{aligned}
f_1&=\frac{1}{8\pi^2}\frac{1+i}{16\kappa^3}\int_\rho^\infty\sigma^2\log|\sigma|\,e^{-\kappa\sigma(1+i)}\,d\sigma\\
f_2&=\frac{1}{8\pi^2}\frac{1-i}{16\kappa^3}\int_\rho^\infty\sigma^2\log|\sigma|\,e^{-\kappa\sigma(1-i)}\,d\sigma\\
f_3&=\frac{1}{8\pi^2}\frac{1+i}{16\kappa^3}\int_{-\infty}^\rho\sigma^2\log|\sigma|\,e^{\kappa\sigma(1+i)}\,d\sigma\\
f_4&=\frac{1}{8\pi^2}\frac{1-i}{16\kappa^3}\int_{-\infty}^\rho\sigma^2\log|\sigma|\,e^{\kappa\sigma(1-i)}\,d\sigma
\end{aligned}\right\} \tag{22}$$

and $\kappa=|p|/\sqrt{2}$. Integration of eqn (21) by parts three times and noting that [15, eqns (3.01), (3.03) and (3.05)]

$$\int_\rho^\infty\frac{e^{-\kappa\sigma(1\pm i)}}{\sigma}\,d\sigma=E_1[\kappa\rho(1\pm i)]\mp\frac{\pi i}{2}(1-\operatorname{sgn}\rho) \tag{23a}$$

$$\int_{-\infty}^\rho\frac{e^{\kappa\sigma(1\pm i)}}{\sigma}\,d\sigma=-E_1[-\kappa\rho(1\pm i)]\pm\frac{\pi i}{2}(1+\operatorname{sgn}\rho) \tag{23b}$$

yields

$$\begin{aligned}
\phi=\frac{1}{128\pi^2\kappa^6}\Big[4\kappa^2\rho^2\log|\rho|&-i\Big\{E_1[\kappa\rho(1+i)]-\frac{\pi i}{2}(1-\operatorname{sgn}\rho)\Big\}e^{\kappa\rho(1+i)}\\
&+i\Big\{E_1[\kappa\rho(1-i)]+\frac{\pi i}{2}(1-\operatorname{sgn}\rho)\Big\}e^{\kappa\rho(1-i)}\\
&-i\Big\{E_1[-\kappa\rho(1+i)]-\frac{\pi i}{2}(1+\operatorname{sgn}\rho)\Big\}e^{-\kappa\rho(1+i)}\\
&+i\Big\{E_1[-\kappa\rho(1-i)]+\frac{\pi i}{2}(1+\operatorname{sgn}\rho)\Big\}e^{-\kappa\rho(1-i)}\Big]
\end{aligned} \tag{24}$$

where E_1 is the exponential integral. Using the series representations for $e^{\pm\kappa\rho(1\pm i)}$ and E_1 [15, eqn (3.06)], and noting that

$$\arg[\kappa\rho(1\pm i)] = \mp\left(\frac{\pi}{4}-\frac{\pi}{2}\operatorname{sgn}\rho\right) \tag{25a}$$

$$\arg[-\kappa\rho(1\pm i)] = \mp\left(\frac{\pi}{4}+\frac{\pi}{2}\operatorname{sgn}\rho\right) \tag{25b}$$

we can obtain

$$\phi = \frac{1}{4\pi^2 p^6}\left[\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(4m+2)!}(p\rho)^{4m+2}\left(\log|\rho| + \log|p| + \gamma - \sum_{s=1}^{4m+2}\frac{1}{s}\right) + \frac{\pi}{4}\operatorname{sgn}(p^2)\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(4m)!}(p\rho)^{4m}\right] \tag{26}$$

where γ is Euler's constant. It should be noted that, in eqn (26), the polynomial terms in ρ of degree not greater than 2 have been deleted because they represent only rigid-body displacements due to application of the tangential forces.

Series representation for Φ

Inserting eqn (26) into eqn (16), we can obtain the series representation for Φ of the form

$$\begin{aligned}\Phi = \frac{1}{2\pi\beta^6}\sum_{n=-\infty}^{\infty}(-1)^n\cos 2n\theta&\left[\sum_{\substack{m=[|n|/2]\\ m\geq 1}}^{\infty}\frac{(-1)^{m-1}}{(2m+1-n)!(2m+1+n)!}\left(\frac{\beta r}{2}\right)^{4m+2}\right.\\ &\times\left\{\Pi_{2m-2,|n|}\left(\log\frac{\beta r}{2}+\gamma-\sum_{s=1}^{2m+1-n}\frac{1}{2s}-\sum_{s=1}^{2m+1+n}\frac{1}{2s}\right)+\sum_{k=-2m+2}^{2m-2}\Pi_{2m-2,|k|}F_{|n-k|}\right\}\\ &\left.+\frac{\pi}{4}\sum_{\substack{m=[(|n|+1)/2]\\ m\geq 2}}^{\infty}\frac{(-1)^{m-1}}{(2m-n)!(2m+n)!}\left(\frac{\beta r}{2}\right)^{4m}\sum_{k=-2m+3}^{2m-3}\Pi_{2m-3,|k|}G_{|n-k|}\right]+\Phi_p\end{aligned} \tag{27}$$

where, with $\tau = a/b$

$$\Phi_p = \frac{1}{24\beta^6}\left(\frac{\beta r}{2}\right)^4\frac{1}{\sqrt{\tau}}\left[\frac{3}{4}+\left(\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right)\cos 2\theta + \frac{1}{4}\left(\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right)^2\cos 4\theta\right] \tag{28a}$$

if $0<\tau\leq 1$

$$\begin{aligned}\Phi_p = -\frac{1}{24\beta^6}\left(\frac{\beta r}{2}\right)^4\left(\frac{1}{1-\tau}\right)&\left[\left(1-\frac{4}{\pi}\arctan\sqrt{|\tau|}\right)\left\{2\cos 2\theta+\left(\frac{1+\tau}{1-\tau}\right)\cos 4\theta\right\}\right.\\ &\left.-\frac{2}{\pi}\sin(\arctan\sqrt{|\tau|})\cos 4\theta\right]\end{aligned} \tag{28b}$$

if $-1\leq\tau\leq 0$†, and $\theta = \arctan(y/x)$, $[N]$ = nearest integer equal to or less than N.

The coefficients Π_{ik}, F_k and G_k in eqn (27) are given respectively by

$$\Pi_{i,|k|} = \sum_{h=0}^{[(i-k)/2]}\binom{i}{|k|+2h}\binom{|k|+2h}{h}\left(\frac{1+\tau}{2}\right)^{i-|k|-2h}\left(\frac{1-\tau}{4}\right)^{|k|+2h}\quad(\text{for } i\geq|k|). \tag{29a}$$

†In the case that $-1\leq\tau\leq 0$, the integral

$$\Phi_p = \frac{1}{16\pi}\frac{1}{4!}\int_\Omega \operatorname{sgn}(p^2)\frac{\rho^4}{p^2}\,d\Omega$$

diverges. However, this divergent integral can be regularized by deleting the terms corresponding to arbitrary rigid-body displacements. To define such a regularization uniquely, we require for $-1\leq\tau\leq 0$ that $\Delta\Delta\Phi$ vanishes at the origin. Physically, this means that the normal deflection due to application of the concentrated normal force becomes zero at the loaded point.

$$\Pi_{i,|k|} = 0 \qquad \text{(for } i < |k|\text{)} \tag{29b}$$

$$F_0 = \log\left(\frac{1+\sqrt{\tau}}{2}\right), \qquad F_k = \frac{(-1)^{k-1}}{2k} \quad (k>0) \qquad \text{(if } 0 < \tau \le 1\text{)} \tag{30a}$$

$$F_0 = \log\left(\frac{\sqrt{(1-\tau)}}{2}\right), \qquad F_k = \frac{(-1)^{k-1}}{2k}\cos 2k\theta^{\dagger} \quad (k>0) \qquad \text{(if } -1 \le \tau < 0\text{)} \tag{30b}$$

$$G_0 = 1 - \frac{4\theta^{\dagger}}{\pi}, \qquad G_k = \frac{2}{\pi}\frac{(-1)^{k-1}}{k}\sin 2k\theta^{\dagger} \quad (k>0) \tag{31}$$

where

$$\theta^{\dagger} = \begin{cases} 0 & \text{(if } 0 < \tau \le 1\text{)} \\ \arctan\sqrt{|\tau|} & \text{(if } -1 \le \tau < 0\text{)} \end{cases} \tag{32}$$

and

$$\binom{i}{j} = \frac{i!}{(i-j)!j!}, \text{ binomial coefficients;} \quad \binom{i}{0} = \binom{0}{0} = 1, \quad 0! = 1.$$

Detailed description of the integral calculation is presented in Ref. [16] and will not be repeated herein.

Representations for derivatives of Φ

Once the fundamental solution Φ has been obtained in the form of eqn (27), it is possible to write down the complete expressions for the stress resultants and displacements for each of the concentrated loading and concentrated heating by use of eqns (7)–(10). In order to evaluate the stress resultants and displacements, we must write down the expressions for the derivatives of Φ. This can of course be accomplished by direct differentiation of eqn (27). An alternative and probably more systematic method is to differentiate ϕ with respect to ρ and then to integrate the results over the unit sphere. For instance

$$D_1{}^p D_2{}^q \Delta^r \Phi = \int_{\Omega} \omega_1{}^p \omega_2{}^q \frac{\mathrm{d}^{p+q+2r}\phi}{\mathrm{d}\rho^{p+q+2r}}\,\mathrm{d}\Omega.$$

The integral in this form can be evaluated systematically in a similar fashion to that used in obtaining Φ. Omitting details of the calculation, we can obtain

$$\begin{aligned} D_1{}^{2i} D_2{}^{2j} \Delta^{t-i-j}\Phi = {} & \frac{\beta^{2t-6}}{2\pi \times 2^{2i+2j}} \sum_{n=-\infty}^{\infty} (-1)^n \cos 2n\theta \sum_{p=-i}^{i}\sum_{q=-j}^{j} (-1)^p \binom{2i}{i-p}\binom{2j}{j-q} \\ & \times \Bigg[\sum_{\substack{m=[(|n|+t)/2] \\ m\ge 1}}^{\infty} \frac{(-1)^{m-1}}{(2m+1-t-n)!(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+2-2t} \\ & \times \left\{ \Pi_{2m-2,|n-p-q|}\left(\log\frac{\beta r}{2} + \gamma - \sum_{s=1}^{2m+1-t-n}\frac{1}{2s} - \sum_{s=1}^{2m+1-t+n}\frac{1}{2s}\right) + \sum_{k=-2m+2+p+q}^{2m-2+p+q} \Pi_{2m-2,|k-p-q|} F_{|n-k|} \right\} \\ & + \frac{\pi}{4} \sum_{\substack{m=[(|n|+t+1)/2] \\ m\ge 2}}^{\infty} \frac{(-1)^{m-1}}{(2m-t-n)!(2m-t+n)!}\left(\frac{\beta r}{2}\right)^{4m-2t} \sum_{k=-2m+3+p+q}^{2m-3+p+q} \Pi_{2m-3,|k-p-q|} G_{|n-k|} \\ & + \frac{1}{2} \sum_{m=1}^{[(|n|+t)/2-1]} (-1)^{m+n+t-1} \frac{(|n|-2m-2+t)!}{(2m+1-t+|n|)!}\left(\frac{\beta r}{2}\right)^{4m+2-2t} \Pi_{2m-2,|n-p-q|} \Bigg] + D_1{}^{2i} D_2{}^{2j} \Delta^{t-i-j}\Phi_p \end{aligned} \tag{33}$$

$$D_1^{2i-1}D_2^{2j-1}\Delta^{t-i-j+1}\Phi = \frac{\beta^{2t-6}}{2\pi\times 2^{2i+2j-3}}\sum_{n=1}^{\infty}(-1)^n \sin 2n\theta \sum_{p=1-i}^{i}\sum_{q=1-j}^{j}(-1)^p\binom{2i-1}{i-p}\binom{2j-1}{j-q}$$

$$\times\Bigg[\sum_{\substack{m=[(n+t)/2]\\ m\geq 1}}^{\infty}\frac{(-1)^{m-1}}{(2m+1-t-n)!(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+2-2t}$$

$$\times\left\{\Pi_{2m-2,|n-p-q+1|}\left(\log\frac{\beta r}{2}+\gamma-\sum_{s=1}^{2m+1-t-n}\frac{1}{2s}-\sum_{s=1}^{2m+1-t+n}\frac{1}{2s}\right)\right.$$

$$\left.+\sum_{k=-2m+1+p+q}^{2m-3+p+q}\Pi_{2m-2,|k-p-q+1|}F_{|n-k|}\right\}$$

$$+\frac{\pi}{4}\sum_{\substack{m=[(n+t+1)/2]\\ m\geq 2}}^{\infty}\frac{(-1)^{m-1}}{(2m-t-n)!(2m-t+n)!}\left(\frac{\beta r}{2}\right)^{4m-2t}$$

$$\times\sum_{k=-2m+2+p+q}^{2m-4+p+q}\Pi_{2m-3,|k-p-q+1|}G_{|n-k|}$$

$$+\frac{1}{2}\sum_{m=1}^{[(n+t)/2-1]}(-1)^{m+n+t-1}\frac{(n-2m-2+t)!}{(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+2-2t}\Pi_{2m-2,|k-p-q+1|}\Bigg]$$

$$+D_1^{2i-1}D_2^{2j-1}\Delta^{t-i-j+1}\Phi_p \tag{34}$$

$$D_1^{2i-1}D_2^{2j}\Delta^{t-i-j}\Phi = \frac{\beta^{2t-7}}{2\pi\times 2^{2i+2j-2}}\sum_{n=1}^{\infty}(-1)^n \cos(2n-1)\theta \sum_{p=1-i}^{i}\sum_{q=-j}^{j}(-1)^p\binom{2i-1}{i-p}\binom{2j}{j-q}$$

$$\times\Bigg[\sum_{\substack{m=[(n+t-1)/2]\\ m\geq 1}}^{\infty}\frac{(-1)^{m-1}}{(2m+2-t-n)!(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+3-2t}$$

$$\times\left\{\Pi_{2m-2,|n-p-q|}\left(\log\frac{\beta r}{2}+\gamma-\sum_{s=1}^{2m+2-t-n}\frac{1}{2s}-\sum_{s=1}^{2m+1-t+n}\frac{1}{2s}\right)\right.$$

$$\left.+\sum_{k=-2m+2+p+q}^{2m-2+p+q}\Pi_{2m-2,|k-p-q|}F_{|n-k|}\right\}$$

$$+\frac{\pi}{4}\sum_{\substack{m=[(n+t)/2]\\ m\geq 2}}^{\infty}\frac{(-1)^{m-1}}{(2m+1-t-n)!(2m-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+1-2t}$$

$$\times\sum_{k=-2m+3+p+q}^{2m-3-p-q}\Pi_{2m-3,|k-p-q|}G_{|n-k|}$$

$$+\frac{1}{2}\sum_{m=1}^{[(n+t-3)/2]}(-1)^{m+n+t}\frac{(n-2m-3+t)!}{(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+3-2t}\Pi_{2m-2,|k-p-q|}\Bigg]$$

$$+D_1^{2i-1}D_2^{2j}\Delta^{t-i-j}\Phi_p \tag{35}$$

$$D_1^{2i}D_2^{2j-1}\Delta^{t-i-j}\Phi = \frac{\beta^{2t-7}}{2\pi\times 2^{2i+2j-2}}\sum_{n=1}^{\infty}(-1)^n \sin(2n-1)\theta \sum_{p=-i}^{i}\sum_{q=1-j}^{j}(-1)^{p-1}\binom{2i}{i-p}\binom{2j-1}{j-q}$$

$$\times\Bigg[\sum_{\substack{m=[(n+t-1)/2]\\ m\geq 1}}^{\infty}\frac{(-1)^{m-1}}{(2m+2-t-n)!(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+3-2t}$$

$$\times\left\{\Pi_{2m-2,|n-p-q|}\left(\log\frac{\beta r}{2}+\gamma-\sum_{s=1}^{2m+2-t-n}\frac{1}{2s}-\sum_{s=1}^{2m+1-t+n}\frac{1}{2s}\right)\right.$$

$$\left.+\sum_{k=-2m+2+p+q}^{2m-2+p+q}\Pi_{2m-2,|k-p-q|}F_{|n-k|}\right\}$$

$$+\frac{\pi}{4}\sum_{\substack{m=[(n+t)/2]\\ m\geq 2}}^{\infty}\frac{(-1)^{m-1}}{(2m+1-t-n)!(2m-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+1-2t}$$

$$\times\sum_{k=-2m+3+p+q}^{2m-3+p+q}\Pi_{2m-3,|k-p-q|}G_{|n-k|}$$

$$\left.+\frac{1}{2}\sum_{m=1}^{[(n+t-3)/2]}(-1)^{m+n+t}\frac{(n-2m-3+t)!}{(2m+1-t+n)!}\left(\frac{\beta r}{2}\right)^{4m+3-2t}\Pi_{2m-2,|n-p-q|}\right]$$

$$+D_1^{2i}D_2^{2j-1}\Delta^{t-i-j}\Phi_p. \quad (36)$$

We are now prepared to write down the complete expressions for the stress resultants and displacements for each of the concentrated loading and concentrated heating, but since they are rather cumbersome, they will be omitted herein.

NUMERICAL RESULTS AND DISCUSSION

To illustrate the use of the fundamental solution such as given by eqn (27), we have selected the case of the concentrated normal force acting on infinite shells with variable a/b. In the numerical evaluation of the solution, the power series for each harmonic were calculated with the help of an electronic computer. The power series were truncated when succeeding terms progressively decreased in magnitude and each additional term was less than 10^{-6} times the accumulated sum. The convergence of the series is greatly influenced by the value of dimensionless radial co-ordinate βr. For smaller values of βr, the convergence is more rapid. For the largest value of βr considered here, the first 12 harmonics in the Fourier-series and the first 10 terms in the power series were taken to obtain the convergent solution. The computations were executed on FACOM 230–75 at Nagoya University Computation Center.

The results for the stress resultants, stress couples and normal deflections are presented graphically in Figs. 3–14. Figures 3–6 display the distributions of the membrane stress resultants N_x and N_y along $y = 0$ and $x = 0$. Note that the value $y = 0$ represents the line of maximum curvature, while $x = 0$ represents the line of minimum curvature. Although N_x, shown in Figs. 3 and 4, does not appear particularly sensitive to shell form, the distribution of N_y is significantly different for various ratios of a/b. As might be expected, N_y for shells of negative Gaussian curvature is tensile near the origin, while it is compressive for shells of positive or zero Gaussian curvature.

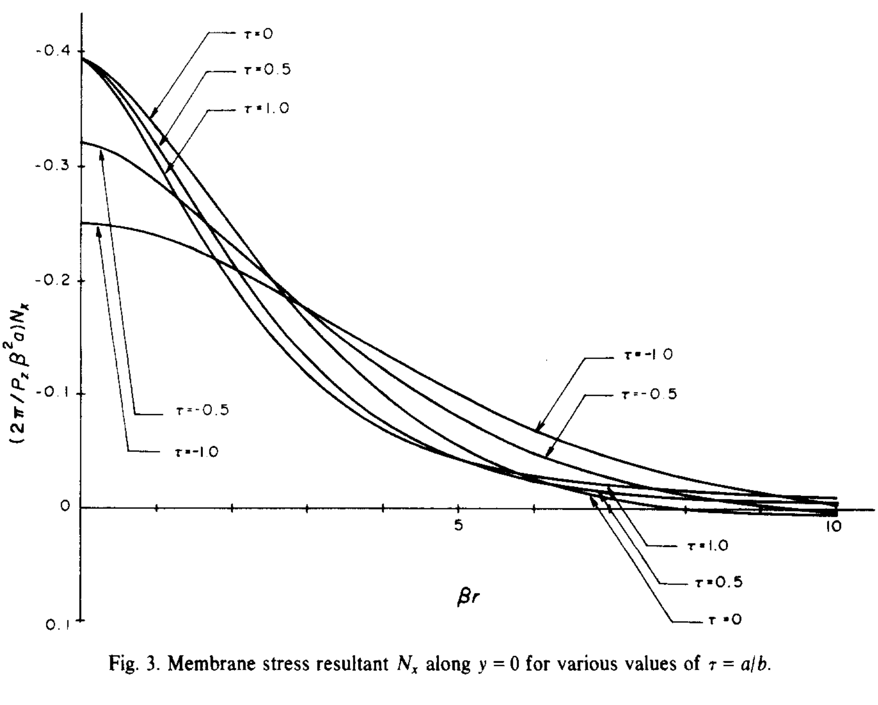

Fig. 3. Membrane stress resultant N_x along $y = 0$ for various values of $\tau = a/b$.

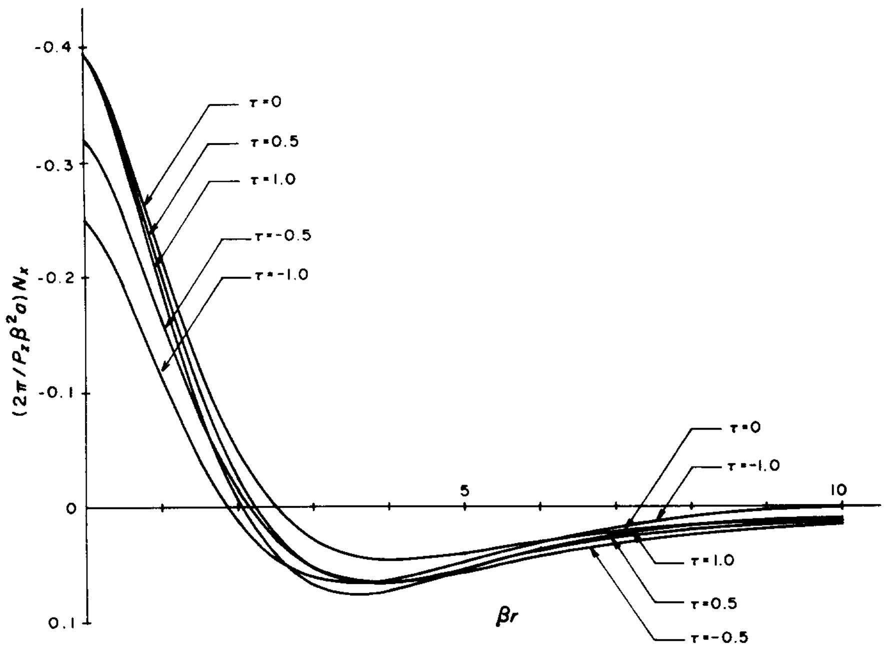

Fig. 4. Membrane stress resultant N_x along $x = 0$ for various values of $\tau = a/b$.

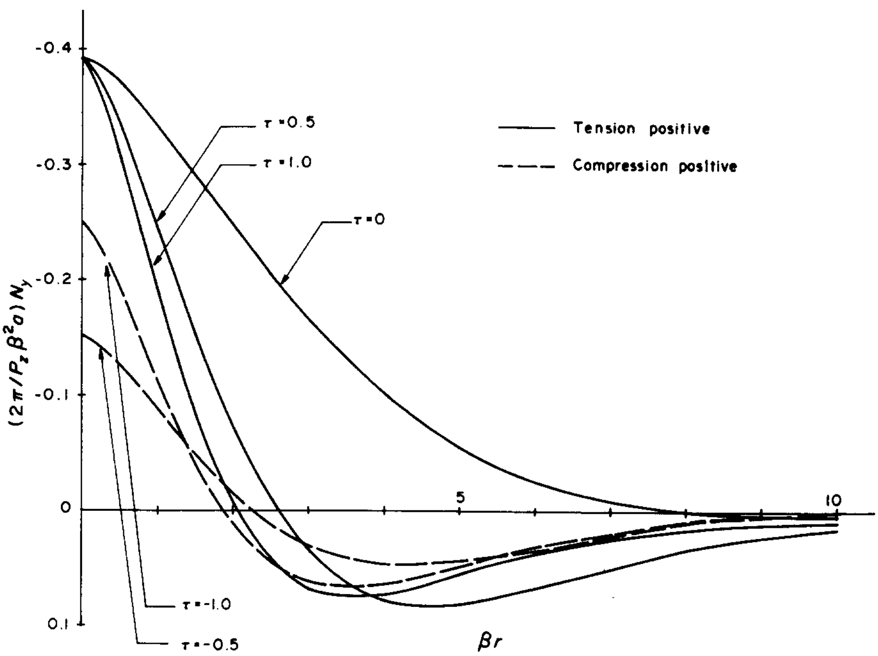

Fig. 5. Membrane stress resultant N_y along $y = 0$ for various values of $\tau = a/b$.

Figures 7–10 demonstrate the distributions of the stress couples M_x and M_y along $y = 0$ and $x = 0$. A value of Poisson's ratio $\nu = 0$ was used in evaluating M_x and M_y. We see in Figs. 9 and 10 that the pattern of the distribution of M_y is quite similar for all the shells. However, the influence of shell geometry is felt significantly in the values of M_x. It should be noted that M_x for the cylinder does not decay so rapidly in the direction of the generator $x = 0$ as in the direction of the circular arc $y = 0$. The distributions of the transverse shear forces Q_x and Q_y, shown in Figs. 11 and 12, are also quite similar for all the ratios of a/b.

The normal deflection w, shown in Figs. 13 and 14, indicates strong depèndence upon the shell geometry. Note that for convenience a constant has been added to the normal deflection of the shell of zero or negative Gaussian curvature so that the displacement at infinity becomes

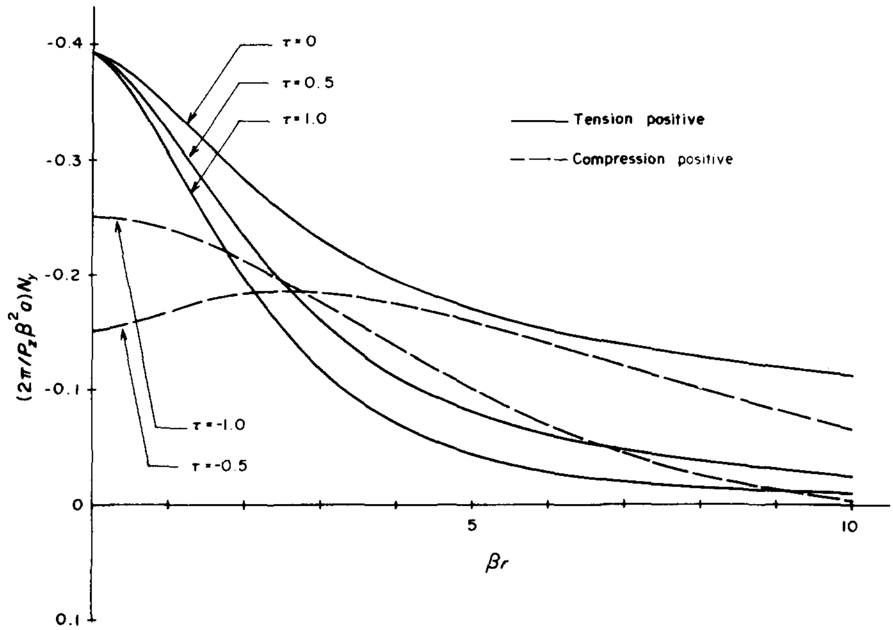

Fig. 6. Membrane stress resultant N_y along $x = 0$ for various values of $\tau = a/b$.

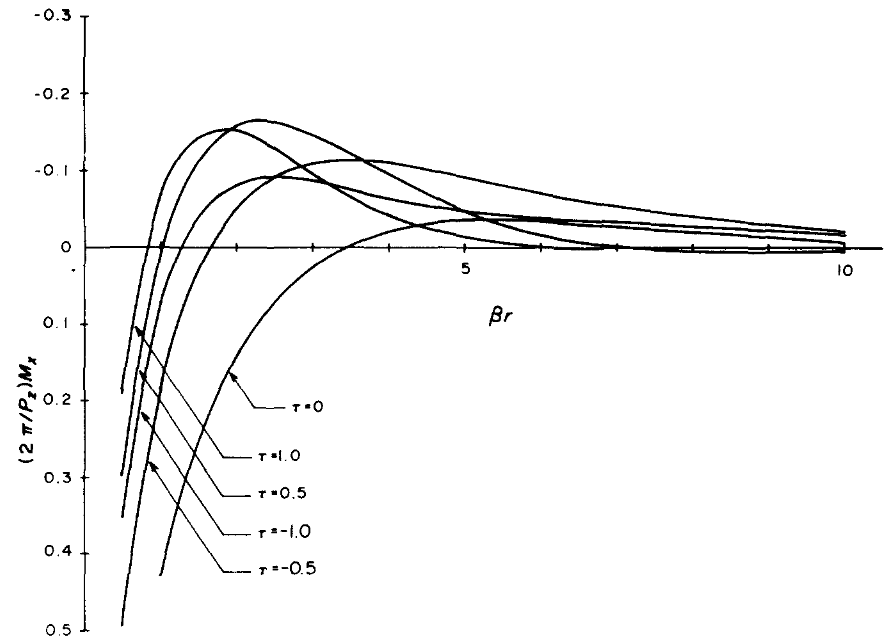

Fig. 7. Stress couple M_x along $y = 0$ for various values of $\tau = a/b$.

zero. (Herein, a point $\beta x = \beta y = 10$ is chosen for convenience as an infinite point). Although the deflection of the shell of positive Gaussian curvature is highly localized near the origin, the deflection of the shell of zero or negative Gaussian curvature no longer vanishes at a distance from the load point. It should be noted that the deflection of the cylinder decays very slowly in the direction of the generator $x = 0$, and the influence of the finite boundaries must be considered.

CONCLUSIONS

A series representation for the fundamental solution of the shallow shell equations has been obtained in the present paper. From this solution we can produce the singular solutions which

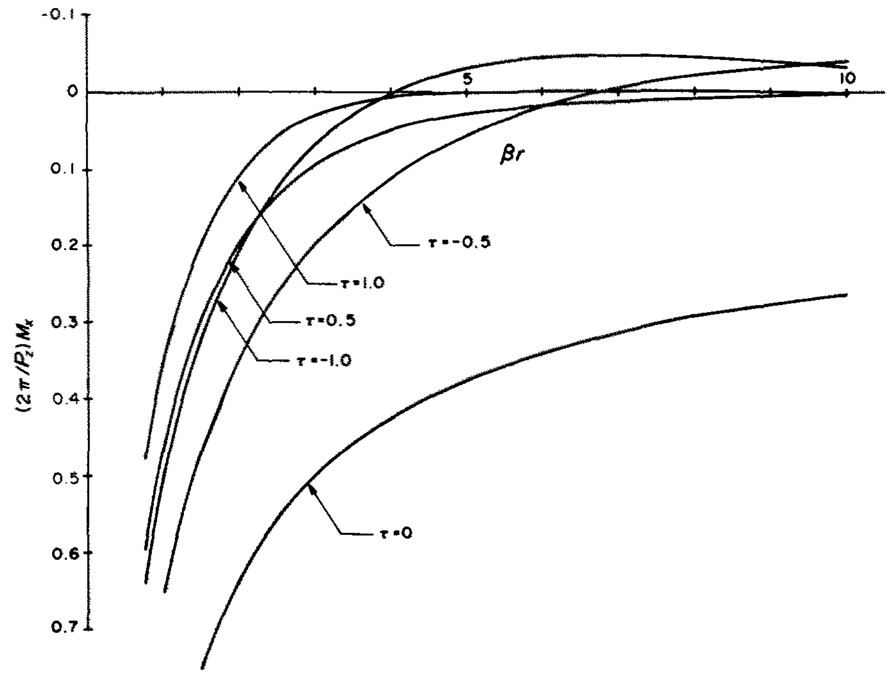

Fig. 8. Stress couple M_x along $x = 0$ for various values of $\tau = a/b$.

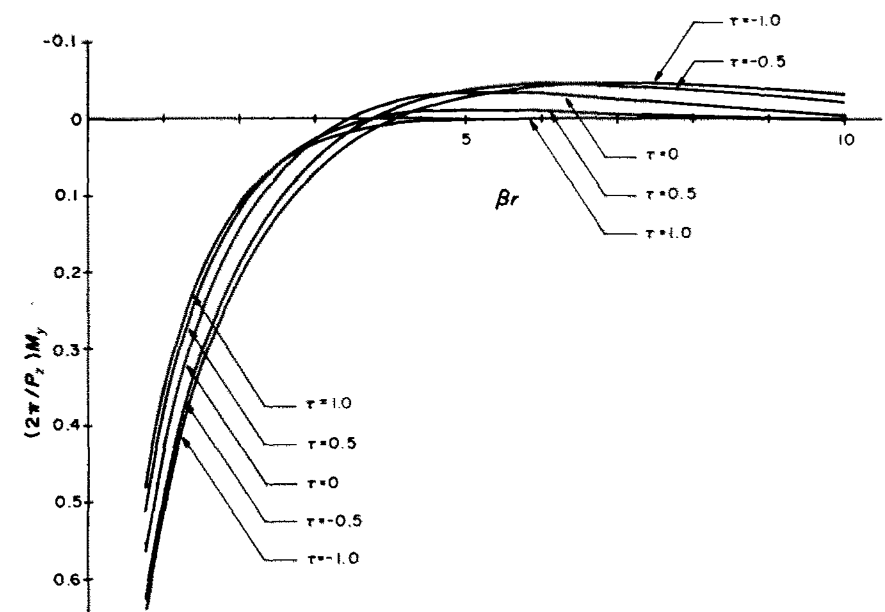

Fig. 9. Stress couple M_y along $y = 0$ for various values of $\tau = a/b$.

correspond to concentrated forces, couples and thermal hot spots applied to a shallow shell with an arbitrary quadratic middle surface. The solutions converge for the entire range of the Gaussian curvature. The use of the fundamental solution obtained here has been illustrated for the case of the concentrated normal force acting on infinite shells having positive, zero or negative Gaussian curvature, and detailed graphical results have been presented for the stress resultants and deflections. The results show that the effect of the concentrated load is felt over a wider region in shells of zero or negative Gaussian curvature compared to shells of positive Gaussian curvature. The fundamental solution presented here will be useful not only to

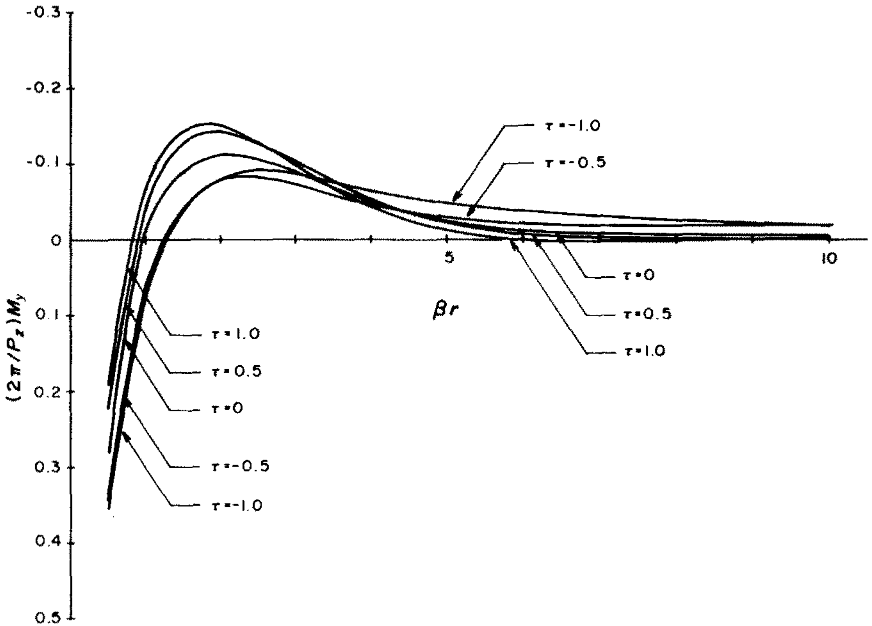

Fig. 10. Stress couple M_y along $x = 0$ for various values of $\tau = a/b$.

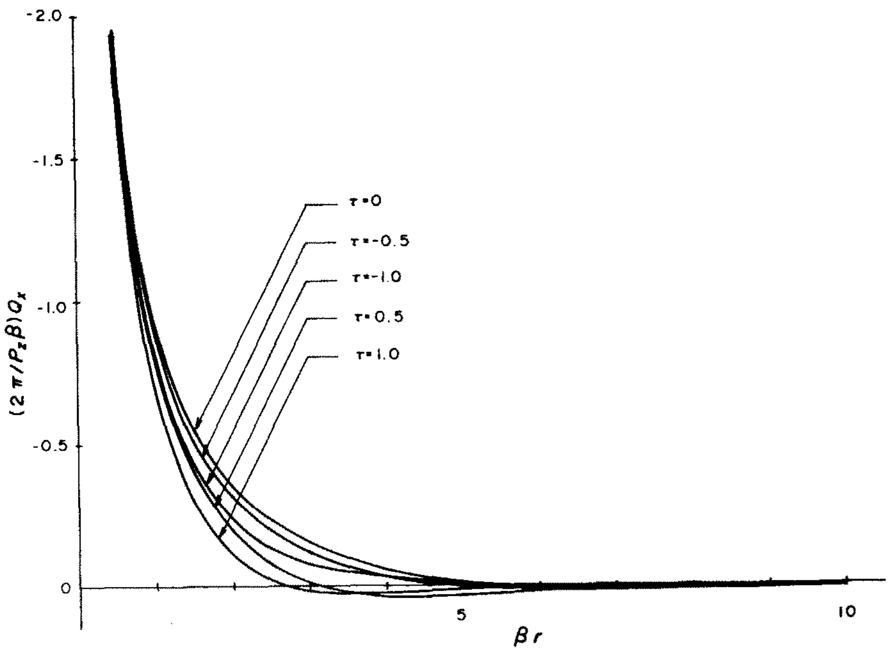

Fig. 11. Transverse shear force Q_x along $y = 0$ for various values of $\tau = a/b$.

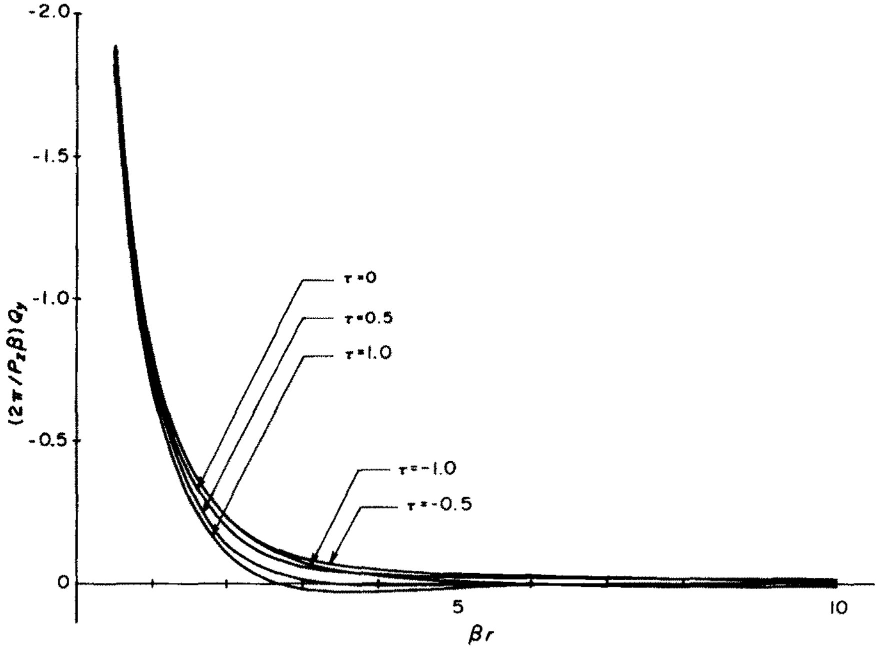

Fig. 12. Transverse shear force Q_y along $x = 0$ for various values of $\tau = a/b$.

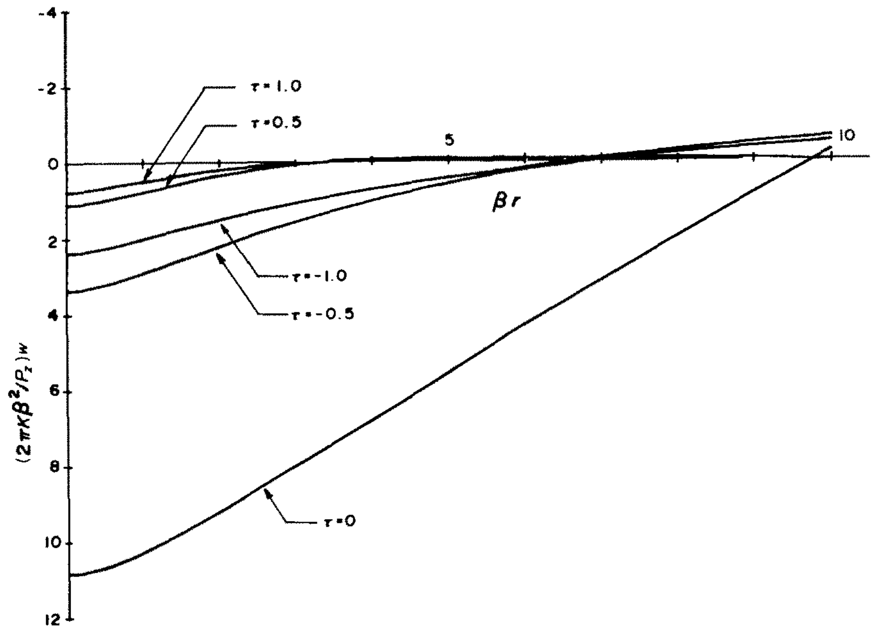

Fig. 13. Normal deflection w along $y = 0$ for various values of $\tau = a/b$.

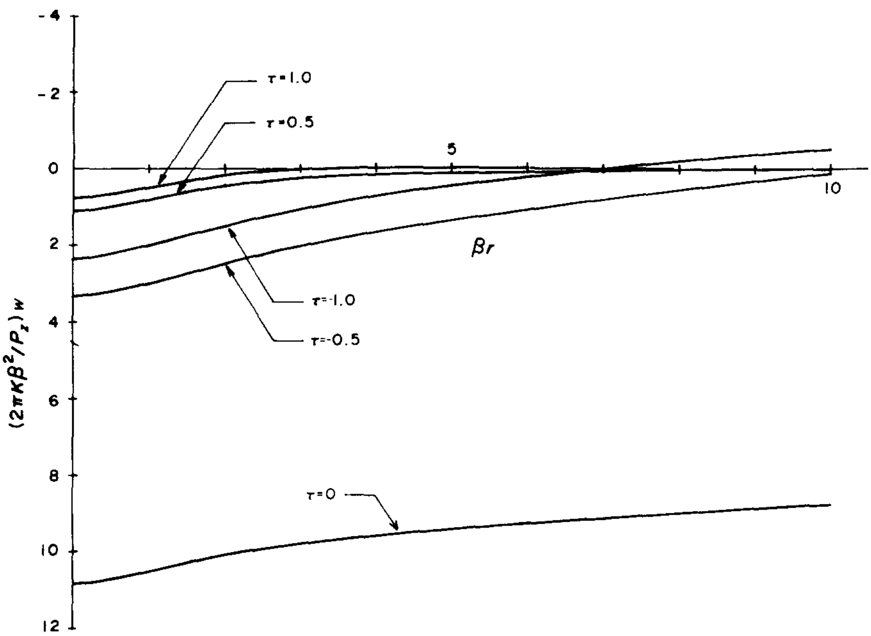

Fig. 14. Normal deflection w along $x = 0$ for various values of $\tau = a/b$.

investigate the problem of concentrated loading or concentrated heating such as discussed in the present paper, but also to study crack or cutout problems in the shallow shell[17].

Note added in proof—After completion of the present work, the authors have learned of two papers by Simmonds and Tropf[18] and Simmonds and Bradley[19], in which the inverse Fourier transforms of the fundamental solutions are presented explicitly for a shallow hyperbolic paraboloid and for a shallow shell with an arbitrary quadratic middle surface. The fundamental solution obtained in the present paper is identical to that presented in the Simmonds and Bradley's paper. However, the expressions for the derivatives of the fundamental solution and the subsequent computations of the stress resultants and deflections for infinite shells subjected to a concentrated normal force are new results not contained in the Simmonds *et al.* paper.

Acknowledgements—The authors wish to express their gratitude to Prof. Emeritus Y. Yokoo at Kyoto University for his valuable suggestion and discussion during the course of the present work. Sincere thanks are due to Mr. K. Nozaki for his cooperation in preparing the manuscript.

REFERENCES

1. E. Reissner, Stresses and small displacements of shallow spherical shells—Parts I and II. *J. Math. Phys.* **25**, 80–85, 279–300 (1947).
2. G. N. Chernyshev, On the action of concentrated forces and moments on an elestic thin shell of arbitrary shape. *J. Appl. Math. Mech. (PMM)* **27**, 172–184 (1963).
3. S. Lukasiewicz, Concentrated loads on shallow spherical shells. *Q. J. Mech. Appl. Math.* **20**, 293–305 (1967).
4. W. Flügge and D. A. Conrad, Thermal singularities for cylindrical shells. *Proc. 3rd U.S. Nat. Congr. Appl. Mech.* pp. 321-328. Springer, Berlin (1958).
5. J. L. Sanders Jr. and J. G. Simmonds, Concentrated forces on shallow cylindrical shells. *J. Appl. Mech., Trans ASME* **37**, 367-373 (1970).
6. K. Foresberg and W. Flügge, Point load on a shallow elliptic paraboloid. *J. Appl. Mech., Trans. ASME* **33**, 575-585 (1966).
7. W. Flügge and R. E. Elling, Singular solutions for shallow shells. *Int. J. Solids Structures*, **8**, 227-247 (1972).
8. R. E. Elling, Concentrated loads applied to shallow shells. *J. Engng Mech. Div. ASCE* **99**, 319-330 (1973).
9. J. L. Sanders Jr., Singular solutions to the shallow shell equations. *J. Appl. Mech., Trans. ASME* **37**, 361-366 (1970).
10. S. Lukasiewicz, Concentrated loadings on shells. *Proc. 11th Int. Cong. Appl. Mech.*, pp. 277-282. Springer, Berlin (1964).
11. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. 1. Academic Press, New York (1964).
12. F. John, *Plane Waves and Spherical Means Applied to Partial Differential Equations.* Interscience, London (1955).
13. A. Jahanshahi, Some notes on singular solutions and the Green's functions in the theory of plates and shells. *J. Appl. Mech., Trans. ASME* **31**, 441-446 (1964).
14. See for instance, C. R. Wylie Jr., *Advanced Engineering Mathematics*, 3rd Edn. McGraw-Hill, New York (1966).
15. F. W. J. Olver, *Asymptotic and Special Functions.* Academic Press, New York (1974).
16. T. Matsui, Methods and solutions of stress and stability analyses of shell structures. Dissertation submitted to Kyoto University in partial fulfilment of the requirements for the degree of Doctor of Engineering (1977).
17. J. L. Sanders Jr., Cutouts in Shallow Shells. *J. Appl. Mech., Trans. ASME* **37**, 374–383 (1970).
18. J. Simmonds and C. Tropf, The fundamental (normal point load) solution for a shallow hyperbolic paraboloidal shell. *SIAM J. Appl. Math.* **27**, 102-120 (1974).
19. J. Simmonds and M. Bradley, The fundamental solution for a shallow shell with an arbitrary quadratic midsurface. *J. Appl. Mech., Trans. ASME* **43**, 286–290 (1976).